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1994 J. Phys. A: Math. Gen. 27 4805

(http://iopscience.iop.org/0305-4470/27/14/011)

J. Phys. A: Math. Gen. 27 (1994) 4805–4829. Printed in the UK

Noise, dissipation and the classical limit in the quantum kicked-rotator problem

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Received 18 February 1994, in final form 17 May 1994

Abstract. Two models for the investigation of the quantum damped kicked-rotator problem are introduced and analysed in a unified fashion. For the first model we follow the Caldeira–Leggett approach while the second constitutes a simplification of the Dittrich–Graham model. These models enable one to investigate the effects of noise and dissipation for systems that exhibit chaos in the classical limit and quantum localization otherwise. The rotator is coupled via its angle coordinate to a heat bath that is held at an arbitrary temperature. Noise time-autocorrelations which may arise from such coupling and the validity of the Markovian approximation are discussed.

1. Introduction

The kicked rotator constitutes a prototype system for the investigation of Hamiltonian chaos [1]. Quantum mechanically, the chaotic nature of the dynamics is suppressed [2] due to localization [3]. It has been found by Ott, Antonsen and Hanson [4] that uncorrelated white noise destroys coherence and hence localization. However, if noise arises from the coupling to a heat bath then a more detailed treatment is desired. Such a treatment should take into account two effects. One is noise autocorrelations which are expected at low temperature [5]. The other is friction which results in damping and dissipation of energy. While noise results in recovery of diffusion, dissipation of energy tends to balance it and a steady state is reached. Similar interplay of noise and dissipation is found in the study of Zener dynamics [6].

Dittrich and Graham (DG) [7, 8] introduced a model for the investigation of the combined effect of noise and dissipation in the damped quantum kicked-rotator (QKR) problem. Using the master-equation approach [9] they were able to compute the quantum mechanical time evolution of the system. Taking the limit of $\hbar \to 0$ they were also able to consider the semiclassical limit which led them to the following conjecture: regardless of the quantization scheme 'in the semiclassical limit, dissipative quantum maps reduce to the classical maps with additional Gaussian noise terms determined by quantum theory'. However, this important pioneering work left open the following questions.

(i) The master-equation approach of this work involves a Markovian treatment of the dynamics [5]. Consequently, long-range noise time-autocorrelations, which may arise, are automatically ignored.

(ii) The effect of finite temperature has not been explored since a zero-temperature heat bath has been assumed.

(iii) The relation of DGs results to the earlier work by Ott *et al* [4] has not been demonstrated.

0305-4470/94/144805+25$19.50

(iv) The classical limit of the DG model is not explicit and has not been investigated.

(v) The implications of the DG conjecture are not clear. Both the DG model and the master-equation approach originated from atomic physics [9]. In the latter context the classical limit is usually of less interest. Evidently, this is not the case in the field of 'quantum chaos' where the fingerprint of the classical limit on the quantum dynamics is considered to be a main issue. Indeed, a different kind of model and analysis are therefore desired.

The main complication that arises once one is interested in coupling a rotator to a bath is how to formulate a coupling scheme that does not ignore the natural periodicity of its angle variable. Otherwise, if one replaces the angle by an extended coordinate, one obtains a new, different problem which has been investigated and named 'the quantum kicked-*particle* problem' in which it was possible there to use the Caldeira–Leggett (CL) coupling scheme [11] which is linear in the position variable. The latter has been used to treat the damped-particle [12] and the damped-oscillator [13] problems and later on was also applied to investigate time-dependent problems [5]. However, it turned out [10] that the quantum kicked-*particle* problem has quite unique features that are not shared by the QKR problem. In particular there is a stronger sensitivity to noise due to a spreading mechanism for the destruction of coherence. Thus, we again face the complication of how to formulate an appropriate coupling scheme which does not ignore the periodicity of the position variable. One strategy is simply to introduce the coupling via the momentum coordinate. Such a model has been introduced [14] and the effect of low-temperature correlated noise on coherence has been investigated. However, it was demonstrated that friction in this latter model does not result in dissipation of energy and therefore this model is also inappropriate for the investigation of the *damped*-QKR problem.

The purpose of the present work is to analyse, in a unified way, two models for the investigation of the damped-QKR problem. The first model is obtained via modification of the standard CL model. The second constitutes a simplification of the DG model. Besides dealing with questions that were introduced in a preceding paragraph we shall also consider the following approximations:

(i) a semiclassical treatment of the dynamics;

(ii) replacement of the bath by a c-number noise source; and

(iii) Markovian treatment of the dynamics.

By 'classical treatment of the dynamics' we mean that the rotator can be treated as a classical object. This does not mean that the quantum nature of the bath may be ignored. Thus we are able to distinguish between quantum effects that originate in the bath (and are therefore model-dependent) and quantal effects that are associated with the quantum nature of the rotator itself. This distinction is crucial in order to resolve such an ambiguity in the DG conjecture.

Replacement of the bath by a c-number noise source is expected to be legitimate on a time scale which is much shorter than the relaxation time. Within the framework of the Markovian treatment, noise time-autocorrelations that may arise due to the quantum nature of the bath are ignored. In a classical treatment of chaotic dynamics this should not bother us since, due to the exponential instability of the phase-space trajectories, we expect no memory for long-range noise time-autocorrelations. In the quantum mechanical problem, the situation is quite different due to long-range dynamical correlations [15, 10]. The interplay of the latter with the former may lead to either enhancement or reduction of the induced diffusion [15, 10, 14].

The outline of the paper is as follows. In sections 2 and 3 the 'ohmic model' and the 'simplified Dittrich–Graham model' are introduced and a classical treatment of the

dynamics is considered. Diffusion in the quantum kicked-rotator problem is analysed in section 4 where the combined effect of noise and dissipation is also discussed. Finally, the conclusions are summarized in section 5. The four appendices at the end of the paper include the details of some calculations.

2. The ohmic model

Consider a particle that is free to move in one dimension and whose unperturbed Hamiltonian may be time-dependent

$$\hat{\mathcal{H}}_0 = \tfrac{1}{2}\hat{p}^2 + V(\hat{x};t) \tag{2.1}$$

where $\hat{x}$ and $\hat{p}$ are conjugate coordinates $[\hat{x},\hat{p}] = i\hbar$. A bath is defined by the Hamiltonian

$$\hat{\mathcal{H}}_{\text{bath}} = \sum_\alpha \frac{\hat{p}_\alpha^2}{2m_\alpha} + \tfrac{1}{2}m_\alpha\omega_\alpha^2\hat{q}_\alpha^2 \tag{2.2}$$

with $[\hat{q}_\alpha,\hat{p}_\alpha] = i\hbar$. The simplest Hamiltonian for the investigation of the damped-particle problem has been proposed by CL [11–13], namely

$$\mathcal{H} = \hat{\mathcal{H}}_0 + \sum_\alpha \frac{\hat{p}_\alpha^2}{2m_\alpha} + \tfrac{1}{2}m_\alpha\omega_\alpha^2\left(\hat{q}_\alpha - \frac{C_\alpha}{m_\alpha\omega_\alpha^2}\hat{x}\right)^2 \tag{2.3}$$

where C_α are coupling constants. Note that the coupling is linear, namely

$$\mathcal{H}_{\text{int}} = -\hat{x}\sum_\alpha C_\alpha\hat{q}_\alpha \tag{2.4}$$

and that the particle experiences the same environment irrespective of its spatial position. The bath is characterized by the spectral function

$$J(\omega) \equiv \frac{\pi}{2}\sum_\alpha \frac{C_\alpha^2}{m_\alpha\omega_\alpha}\delta(\omega-\omega_\alpha). \tag{2.5}$$

The significance of this spectral function will be apparent in the following.

Consider now a rotator whose unperturbed Hamiltonian is still (2.1); however, periodic boundary conditions are imposed on the interval $[0, 2\pi]$. A heuristic visualization of the system is to consider a particle which is free to move in a one-dimensional ring. The simplest choice for the interaction term $\mathcal{H}_{\text{int}}$ that constitutes a linear coupling scheme is to replace $\hat{x}$ in (2.4) by a periodic function of $\hat{x}$, e.g. $\sin(\hat{x})$. However, for such a choice the particle experiences a non-homogeneous environment, i.e. in different parts of the ring the local environment is different. In order to overcome this problem we propose to take

$$\hat{\mathcal{H}}_{\text{int}} = -\sum_\alpha C_\alpha\hat{q}_\alpha\sqrt{2}\sin(\hat{x}+\varphi_\alpha) \tag{2.6}$$

with phases φ_α such that

$$\frac{\pi}{2}\sum_\alpha \frac{C_\alpha^2}{m_\alpha\omega_\alpha}\delta(\omega-\omega_\alpha)\delta(\varphi-\varphi_\alpha) = \frac{1}{2\pi}J(\omega) \tag{2.7}$$

where $J(\omega)$ is the spectral function (2.5) of the bath. Thus, if we consider a partition of the bath oscillators into subsets of oscillators whose frequencies ω_α are the same, then, within each subset, the φ_α are distributed uniformly. Finally, the total Hamiltonian of the system plus the heat bath is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} + \mathcal{H}_{\text{bath}}. \tag{2.8}$$

The quantum state of the particle may be represented by the Wigner function $\rho(x, p)$. The time evolution of the Wigner function for *finite* time and sufficiently small $\hbar$ may be approximated by that of a classical distribution in phase space. We shall now use such an approximation and we shall title it a 'classical treatment of the dynamics'. The latter term implies that the system is considered to be classical while the bath gets full quantum mechanical treatment. The limit $\hbar \to 0$ is not taken. The equations of motion of classical points that form a distribution in phase space are $\dot{x} = p$ and

$$\dot{p} = -V'(x) + \int \mathrm{d}\varphi \sqrt{2}\cos(x + \varphi)\mathcal{F}_\varphi(t) \tag{2.9}$$

where

$$\mathcal{F}_\varphi(t) = \sum_\alpha C_\alpha q_\alpha(t)\delta(\varphi - \varphi_\alpha). \tag{2.10}$$

The variables $q_\alpha(t)$ satisfy the equation

$$m_\alpha \ddot{q}_\alpha(t) + m_\alpha \omega_\alpha^2 q_\alpha(t) = C_\alpha \sqrt{2}\sin(x(t) + \varphi_\alpha) \tag{2.11}$$

which can be solved explicitly, i.e.

$$q_\alpha(t) = q_\alpha(0)\cos(\omega_\alpha t) + \frac{p_\alpha(0)}{m_\alpha \omega_\alpha}\sin(\omega_\alpha t) + \int_{t'=0}^{t} \frac{C_\alpha}{m_\alpha \omega_\alpha}\sin\omega_\alpha(t - t')\sqrt{2}\sin(x(t') + \varphi_\alpha)\,\mathrm{d}t'. \tag{2.12}$$

Substitution of (2.12) into definition (2.10) yields

$$\mathcal{F}_\varphi = F_\varphi^{\text{friction}} + F_\varphi \tag{2.13}$$

with

$$F_\varphi = \sum_\alpha \delta(\varphi - \varphi_\alpha) C_\alpha \left[\cos(\omega_\alpha t)q_\alpha(0) + \sin(\omega_\alpha t)\frac{p_\alpha(0)}{m_\alpha \omega_\alpha}\right] \tag{2.14}$$

while

$$F_\varphi^{\text{friction}} = \int_0^t 2\alpha(t - t')\sqrt{2}\sin(x(t') + \varphi)\,\mathrm{d}t'. \tag{2.15}$$

The response kernel $\alpha(t - t')$ is defined for positive times ($t > t'$) as follows:

$$\alpha(t - t') = \int_0^\infty \frac{\mathrm{d}\omega}{\pi} J(\omega)\sin[\omega(t - t')]. \tag{2.16}$$

The definition of the response kernel for negative times is of no significance. For simplicity, we adopt the convention $\alpha(t-t')\equiv 0$ for $t<t'$.

In order to make further progress, a specification of the initial state of the system plus the bath is needed. We shall assume that initially (at time $t=0$) the system is prepared in some arbitrary quantum state while the bath oscillators are in thermal canonical equilibrium with some reciprocal temperature β. The Wigner-function representation of the probability-density matrix is then (see appendix A)

$$\rho_{t=0}(x,p;q_\alpha,p_\alpha)=\rho_{t=0}(x,p)\prod_\alpha \rho_{\rm eq}(q_\alpha,p_\alpha) \tag{2.17}$$

where

$$\rho_{\rm eq}(q_\alpha,p_\alpha)=\frac{1}{\frac{1}{2}\coth(\frac{1}{2}\beta\hbar\omega_\alpha)}\exp\left[-\beta\left(\frac{\tanh(\frac{1}{2}\beta\hbar\omega_\alpha)}{\frac{1}{2}\beta\hbar\omega_\alpha}\right)\left(\frac{p_\alpha^2}{2m_\alpha}+\tfrac{1}{2}m_\alpha\omega_\alpha^2q_\alpha^2\right)\right]. \tag{2.18}$$

Using (2.18) one obtains the expectation values

$$\left\langle\frac{p_\alpha(0)^2}{2m_\alpha}\right\rangle=\langle\tfrac{1}{2}m_\alpha\omega_\alpha^2q_\alpha(0)^2\rangle=\tfrac{1}{4}\hbar\omega_\alpha\coth(\tfrac{1}{2}\beta\hbar\omega_\alpha) \tag{2.19}$$

and hence it is easily found that $\langle F_\varphi(t)\rangle=0$, while

$$\langle F_\varphi(t)F_{\varphi'}(t')\rangle=\frac{1}{2\pi}\delta(\varphi-\varphi')\phi(t-t') \tag{2.20}$$

where

$$\phi(t-t')=\int_0^\infty\frac{{\rm d}\omega}{\pi}J(\omega)\hbar\coth(\tfrac{1}{2}\beta\hbar\omega)\cos[\omega(t-t')]. \tag{2.21}$$

Now we may turn back to the equation of motion (2.9), substitute the expression (2.13) and use the results (2.15) and (2.20) in order to cast the equation into Langevin's form, namely

$$\dot p=-V'(x)+F^{\rm friction}+F(t). \tag{2.22}$$

The friction term $F^{\rm friction}$ originates from $F_\varphi^{\rm friction}$ and takes the form

$$F^{\rm friction}=-\int_0^t 2\alpha(t-t')\ \sin[x(t)-x(t')]\,{\rm d}t'. \tag{2.23}$$

The noise term $F(t)$ originates from $F_\varphi(t)$ and satisfies $\langle F(t)\rangle=0$, while locally

$$\langle F(t)F(t')\rangle=\phi(t-t'). \tag{2.24}$$

The Langevin equation (2.22) together with (2.23) and (2.24) constitute a complete description of the reduced dynamical behaviour of the system on a time scale such that quantum to classical correspondence is expected.

Further simplification of the expression (2.23) for the friction term is possible if $\alpha(\tau)$ decays by time scale $\tau_{\rm c}$ which is small compared with the dynamical time scales of the unperturbed problem. The validity of this assumption should be established once we let

$J(\omega)$ have some specific functional form. We may then substitute $x(t) - x(t') = p(t - t')$ into (2.22) and obtain the result

$$F^{\text{friction}} = -\operatorname{sgn}(p) J(\omega = |p|) \tag{2.25}$$

where $J(\omega)$ is the spectral function of the bath. If we further assume that the heat bath is an 'ohmic heat bath' in the sense of CL [11, 13], namely

$$J(\omega) = \eta \omega e^{-\omega/\omega_c} \tag{2.26}$$

then the friction is proportional to velocity

$$F^{\text{friction}} = -\eta p. \tag{2.27}$$

It is assumed that the cutoff ω_c is much larger than all other relevant frequencies of the problem. The noise, using (2.21), has the autocorrelation function [5]

$$\phi(\tau) = \frac{\hbar\eta}{\pi}\left[\frac{\tau_c^2 - \tau^2}{(\tau_c^2 + \tau^2)^2}\right] + 2\frac{\eta}{\beta}\frac{1}{2\pi\hbar\beta}\left[\frac{1}{(\tau/\hbar\beta)^2} - \left(\frac{\pi}{\sinh(\pi\tau/\hbar\beta)}\right)^2\right] \tag{2.28}$$

where $\tau_c = 1/\omega_c$. It has two regimes of behaviour; the short time where

$$\phi(\tau) = -\frac{\hbar\eta}{\pi}\frac{1}{\tau^2} \qquad \text{for} \qquad \tau_c \ll \tau \ll \hbar\beta \tag{2.29}$$

and the long-time regime

$$\phi(\tau) = -2\frac{\eta}{\beta}\frac{2\pi}{\hbar\beta}\exp\left(-2\pi\frac{\tau}{\hbar\beta}\right) \qquad \text{for} \qquad \hbar\beta \ll \tau. \tag{2.30}$$

It satisfies the sum rule

$$\int_{-\infty}^{\infty} \phi(\tau)\,d\tau = 2\frac{\eta}{\beta}. \tag{2.31}$$

In the limit of high temperatures, i.e. if $\hbar\beta$ is smaller then the relevant dynamical time scales, this autocorrelation function may be replaced by the well known classical expression

$$\phi(\tau) = 2\frac{\eta}{\beta}\delta(\tau) \tag{2.32}$$

representing white (uncorrelated) noise.

One may have the incorrect impression that the functional form of the interaction term (2.6) is essential in order to obtain the 'ohmic behaviour' (2.27). We therefore make a digression to show that this is not actually the case. Let us now assume a more general form for the interaction term, namely

$$\mathcal{H}_{\text{int}} = -\sum_{\alpha} C_\alpha q_\alpha u(x + \varphi_\alpha) \tag{2.33}$$

where $u(\varphi)$ is some 2π-periodic function. However, we restrict ourselves to the case of the ohmic bath. The spectral function $J(\omega)$ has the CL form (2.26) and consequently, using (2.16), the response kernel is

$$2\alpha(\tau) = -\eta\frac{\partial}{\partial\tau}\left(\frac{2}{\pi}\,\frac{\tau_c}{\tau_c^2+\tau^2}\right) \qquad \text{for} \qquad 0<\tau \tag{2.34}$$

with $\tau_c = 1/\omega_c$. Assuming τ_c to be shorter than the dynamical time scales of the unperturbed problem one may use instead the expression

$$2\alpha(\tau) = \frac{2}{\pi}\eta\omega_c\delta(\tau) - \eta\delta'(\tau). \tag{2.35}$$

Using the same procedure that previously led to (2.23), one obtains equations (2.6), (2.9), (2.11), (2.12) and (2.15), with the replacement $\sqrt{2}\sin(\,) \to u(\,)$ and $\sqrt{2}\cos(\,) \to u'(\,)$, and consequently

$$F^{\text{friction}} = -\eta\left[\int (u'(\varphi))^2\,\mathrm{d}\varphi\right]\dot{x}. \tag{2.36}$$

The factor in the square brackets can be rescaled to 1 by an appropriate normalization of u with no loss of generality to obtain the desired expression (2.27).

Classical treatment of the dynamics implies treating the system as a classical object while the bath is considered to be a quantal entity. We were able to obtain the reduced equations of motion for the system and to distinguish between 'noise' and 'friction' effects. One may wonder whether a similar (corresponding) reduction is possible within the framework of a full quantum mechanical treatment and, furthermore, whether the distinction between noise and friction is still meaningful. In particular, inspired by the results of the classical treatment, one is interested in the question as to whether the effect of the bath is the same as that of a c-number noise source if friction is ignored. Indeed the Feynman–Vernon formalism [16] enables one to consider the exact reduced dynamics of a system that is coupled to a bath. A detailed presentation of this formalism will not be given in this paper, only some observations which are needed for later discussions are referred to in what follows.

The Hamiltonian of the system combined with the bath is (2.8) with interaction terms given by (2.4) for the CL model and (2.6) for our ohmic model. The system and the bath are assumed to be prepared initially as in (2.17). A path-integral formula enables one to compute the reduced propagator $\mathcal{K}$ of the system. The time evolution for an arbitrary preparation is

$$\rho_t(x_t, p_t) = \int \mathcal{K}(x_t, p_t|x_0, p_0)\,\rho_{t=0}(x_0, p_0)\,\mathrm{d}x_0\,\mathrm{d}p_0 \tag{2.37}$$

where $\rho(x, p)$ is the Wigner-function representation of the reduced probability-density matrix of the system. The reduced propagator is found to be a functional of the external potential $V(x, t)$, the response kernel $\alpha(\tau)$ and the noise autocorrelation function $\phi(\tau)$, namely

$$\mathcal{K} = \mathcal{K}[V(x;t), \alpha(\tau), \phi(\tau)]. \tag{2.38}$$

Using (2.38), it is found that if $\alpha(\tau)$ decays on time scale τ_c, which is small compared with the time scales of the unperturbed problem, and if the noise is white (i.e. $\phi(\tau)$ is a

δ-function) then a Markovian treatment of the dynamics is exact. That is the propagator $\mathcal{K}$ can be factorized and be written as a convolution of shorter time steps. For example

$$\mathcal{K}(x_{t_2}, p_{t_1}|x_{t_0}, p_{t_0}) = \int \mathcal{K}(x_{t_1}, p_{t_2}|x_{t_1}, p_{t_1})\mathcal{K}(x_{t_1}, p_{t_1}|x_{t_0} p_{t_0})\, \mathrm{d}x_{t_1}\, \mathrm{d}p_{t_1}. \tag{2.39}$$

More details may be found in [5] where the validity of the Markovian treatment is discussed. A further observation [16, 5] is that the true reduced propagator (2.38) may be obtained using the following prescription:

(i) add an appropriate driving force F to the external potential $V(x; t)$;

(ii) set $\phi(\tau)$ in equation (2.38) to zero;

(iii) compute the resultant fictitious 'noiseless' propagator $\mathcal{K}_F$; and

(iv) to obtain the true propagator $\mathcal{K}$, average $\mathcal{K}_F$ over realizations of F taken from a Gaussian ensemble.

Thus the 'noiseless' propagator $\mathcal{K}_F$ corresponds to a specific realization of F, while the true propagator $\mathcal{K}$ corresponds to the influence of a c-number noise source that is represented by F. To illustrate this prescription let us refer first to the CL model. The proper definition of the 'noiseless' propagator for the CL model is

$$\mathcal{K}_F \equiv \mathcal{K}[V(x; t) - xF(t)\alpha(\tau), 0] \tag{2.40}$$

where $F(t)$ is a real function of time. To obtain $\mathcal{K}$, the 'noiseless' propagator $\mathcal{K}_F$ should be averaged over realizations of $F(t)$ such that $\langle F(t)\rangle = 0$ and

$$\langle F(t)F(t')\rangle = \phi(t - t') \tag{2.41}$$

where $\langle \ldots \rangle$ denotes here the average that is taken over the realizations of $F(t)$. Since $\mathcal{K}_F$ corresponds to the time evolution in the presence of friction, it follows that, as long as the friction effect is negligible, the Hamiltonian (2.3) may be replaced by

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 - \hat{x}F(t) \tag{2.42}$$

where average over realization of $F(t)$ is implicit. This argument can also be extended for the purpose of investigation of our ohmic model. Namely, one may replace the Hamiltonian (2.8) with (2.6) by

$$\mathcal{H} = \mathcal{H}_0 - \int \mathrm{d}\varphi\, F_\varphi(t)\sqrt{2}\sin(\hat{x} + \varphi) \tag{2.43}$$

where $F_\varphi(t)$ satisfies $\langle F_\varphi(t)\rangle = 0$ and

$$\langle F_\varphi(t)F_{\varphi'}(t')\rangle = \frac{1}{2\pi}\delta(\varphi - \varphi')\phi(t - t'). \tag{2.44}$$

Thus, we conclude the following: the effect of the bath may be represented by the combination of friction and a fluctuating c-number classical force. The time-autocorrelations of the latter are determined by the nature of the bath and the coupling scheme through the single spectral function $J(\omega)$. In particular:

(i) the distinction between noise and friction, which is modelled by the classical Langevin equation, is also a natural consequence of the quantum mechanical treatment;

(ii) on time scales such that the friction effect is negligible, the effect of the bath is the same as that of a c-number noise source;

(iii) Markovian treatment of the dynamics is valid provided the noise is white. Condition (iii) is met only at high temperatures.

3. The Dittrich–Graham model and its simplification

Dittrich–Graham (DG) [7, 8] have considered the quantum kicked rotator coupled to a zero-temperature bath. The unperturbed Hamiltonian of the rotator is of the general form (2.1) and the bath is defined via

$$\hat{\mathcal{H}}_{\text{bath}} = \sum_{\alpha} \hbar\omega_\alpha \hat{a}_\alpha^+ \hat{a}_\alpha. \tag{3.1}$$

The interaction term is

$$\hat{\mathcal{H}}_{\text{int}} = \sum_{\alpha} \hbar g_\alpha (\hat{\Gamma}\hat{a}_\alpha^\dagger + \hat{\Gamma}^\dagger \hat{a}_\alpha) \tag{3.2}$$

where the g_α are coupling constants and

$$\hat{\Gamma} \equiv \sum_{n=0}^{\infty} \sqrt{n}(|n-1\rangle\langle n| + |-n+1\rangle\langle -n|). \tag{3.3}$$

The eigenstates $|n\rangle$ of $\hat{p}$ are such that $\hat{p}|n\rangle = \hbar n|n\rangle$. The bath, which has been considered only for zero temperature, is chosen so that

$$\sum_{\alpha} g_\alpha^2 \delta(\omega - \omega_\alpha) = \frac{1}{2\pi}\eta \tag{3.4}$$

(see [7], equations (3.10)–(3.16)). The disadvantage of this model in the present form is that the classical limit is not explicit. Therefore, one is urged to rewrite it in a somewhat more convenient form. First, one substitutes

$$\hat{a}_\alpha \equiv \left(\frac{m_\alpha\omega_\alpha}{2\hbar}\right)^{1/2} \hat{q}_\alpha + \mathrm{i}\left(\frac{1}{2\hbar m_\alpha\omega_\alpha}\right)^{1/2} \hat{p}_\alpha \tag{3.5}$$

so that the bath Hamiltonian (3.1) takes the standard form (2.2). Then $\hat{\Gamma}$ is expressed via the dynamical variables of the rotator

$$\begin{aligned}\hat{\Gamma} &= \sum_{n=0}^{\infty} \sqrt{n}\left((\mathrm{e}^{-\mathrm{i}\hat{x}}|n\rangle\langle n|) + (\mathrm{e}^{\mathrm{i}\hat{x}}|-n\rangle\langle -n|)\right) \\ &= \mathrm{e}^{-\mathrm{i}\hat{x}}\sqrt{\left|\frac{\hat{p}}{\hbar}\right|}\theta(\hat{p}) + \mathrm{e}^{\mathrm{i}\hat{x}}\sqrt{\left|\frac{\hat{p}}{\hbar}\right|}\theta(-\hat{p})\end{aligned} \tag{3.6}$$

where the operator $\sqrt{|\hat{p}|}$ is defined via its diagonal representation and θ is the step function. The interaction term (3.2) can now be cast into the form

$$\hat{\mathcal{H}}_{\text{int}} = \hat{u}\sum_{\alpha} C_\alpha \hat{q}_\alpha + \hat{v}\sum_{\alpha} \tilde{C}_\alpha \hat{p}_\alpha. \tag{3.7}$$

The operators $\hat{u}$ and $\hat{v}$ are $\sqrt{\hbar}\frac{1}{2}(\Gamma^\dagger + \Gamma)$ and $\sqrt{\hbar}\frac{\mathrm{i}}{2}(\Gamma^\dagger - \Gamma)$, respectively. The coupling constants are $C_\alpha = \sqrt{2m_\alpha\omega_\alpha}g_\alpha$ and $\tilde{C}_\alpha = \sqrt{2/m_\alpha\omega_\alpha}g_\alpha$ and the spectral functions of the bath with respect to the dynamical variables $\hat{u}$ and $\hat{v}$ are, respectively,

$$\frac{\pi}{2}\sum_{\alpha}\frac{C_\alpha^2}{m_\alpha\omega_\alpha}\delta(\omega-\omega_\alpha) = \frac{\pi}{2}\sum_{\alpha} m_\alpha\omega_\alpha\tilde{C}_\alpha^2\delta(\omega-\omega_\alpha) = \tfrac{1}{2}\eta. \tag{3.8}$$

The essential features of the DG model are now transparent: the heat bath is non-ohmic (i.e. its spectral function does not have the CL form (2.26)) and the coupling to the bath is momentum-dependent (i.e. $\hat{u}$ and $\hat{v}$ also depend on the momentum $\hat{p}$, unlike in (2.33)). In order to see the momentum dependence more clearly we introduce for concreteness the explicit expression for $\hat{u}$, namely

$$\hat{u} = \sqrt{\hbar}\tfrac{1}{2}(\Gamma + \Gamma^{\dagger}) = \tfrac{1}{2}(\cos\hat{x}\sqrt{|\hat{p}|} + \sqrt{|\hat{p}|}\cos\hat{x}) + \mathcal{O}(\hbar). \tag{3.9}$$

It will be apparent in the subsequent discussion that the proportionality $\hat{u} \propto \sqrt{|\hat{p}|}$ is crucial in order to obtain friction that is proportional to velocity in the case of coupling to this non-ohmic bath.

We shall introduce now a simplified model that has the same essential features as those of the DG model. The bath is assumed to have the Hamiltonian (2.2) while the interaction term is

$$\hat{\mathcal{H}}_{\text{int}} = -\sum_{\alpha} C_{\alpha}\hat{q}_{\alpha}\tfrac{1}{2}[\sqrt{2}\sin(\hat{x} + \varphi_{\alpha})\sqrt{|\hat{p}|} + \text{HC}]. \tag{3.10}$$

The spectral function of the bath is non-ohmic and has the form

$$J_{\text{DG}}(\omega) = \eta e^{-\omega/\omega_c} \tag{3.11}$$

which is different from the CL form (2.26). The phases φ_{α} are distributed uniformly so that (2.7) is satisfied. The same procedure that led to the Langevin form (2.22) of the classical equation of motion may be used (appendix B) to obtain

$$\begin{aligned} \dot{x} &= p + G^{\text{friction}} + G(t) \\ \dot{p} &= -V'(x) + F^{\text{friction}} + F(t). \end{aligned} \tag{3.12}$$

The friction term that is responsible for the damping effect is

$$F^{\text{friction}} = -\int_0^t 2\alpha(t-t')\sqrt{|p(t)p(t')|}\,\sin[x(t) - x(t')]\,dt' \tag{3.13}$$

instead of (2.23). The kernel $\alpha(t-t')$ is defined as in (2.16) with the appropriate spectral function (3.11). Further simplification is possible if $\tau_c = 1/\omega_c$ is shorter compared with other time scales of the problem, leading to (appendix B)

$$F^{\text{friction}} = -J(\omega = |p|)p = -\eta p \tag{3.14}$$

instead of (2.25). Hence, in spite of the non-ohmic bath (3.11), the friction is indeed proportional to the velocity as for the ohmic model. The expression for the non-generic friction term is found in a similar fashion (appendix B) leading to

$$G^{\text{friction}} = -\frac{\eta}{\pi}\,\text{sgn}(p)\int_0^{\infty}\frac{u}{1+u^2}\cos\left(\frac{p}{\omega_c}u\right)du. \tag{3.15}$$

This friction term diverges in the limit $p/\omega_c \to 0$. This unphysical divergence may be avoided by putting a lower cutoff on the function $J(\omega)$ in the vicinity of $\omega = 0$. Furthermore, we shall immediately see that such a lower cutoff is required to make the model well defined

for finite temperatures. We turn to discuss the noise terms which appear in (3.15). These noise terms (appendix B) satisfy $\langle F(t)\rangle = \langle G(t)\rangle = 0$ and

$$\langle F(t)F(t')\rangle = |p|\phi_{\mathrm{DG}}(t-t') \tag{3.16a}$$

$$\langle G(t)G(t')\rangle = \frac{1}{4|p|}\phi_{\mathrm{DG}}(t-t') \tag{3.16b}$$

$$\langle F(t)G(t')\rangle = 0. \tag{3.16c}$$

Note that (3.16c) is not self-evident. The autocorrelation function $\phi_{\mathrm{DG}}(t-t')$ is obtained by substitution of the non-ohmic spectral function (3.11) in the usual definition (2.21). This autocorrelation function is ill defined at finite temperature since it then constitutes a Fourier transform of 1/frequency fluctuations spectrum, i.e.

$$\phi_{\mathrm{DG}}(\omega) = 2\frac{\eta}{\beta}\left|\frac{1}{\omega}\right| \qquad \text{for } |\omega| \ll \frac{1}{\hbar\beta}. \tag{3.17}$$

Consequently, the variance of the noise which is given by $\int_0^\infty \phi(\omega)\mathrm{d}\omega/\pi$ is infinite. A well defined expression for $\phi_{\mathrm{DG}}(t-t')$ is obtained only at zero temperature:

$$\phi_{\mathrm{DG}}(t-t') = \hbar\eta\delta(t-t') \tag{3.18}$$

which is a white uncorrelated noise unlike the ohmic model where (2.29) applies.

Dittrich and Graham ([7], equation (4.9)) have considered a map that constitutes the discrete-time version of the Langevin equation (3.12). It reads:

$$\begin{aligned} x_t &= x_{t-1} + p_{t-1} + \psi_t \\ p_t &= \lambda p_{t-1} + \eta_t + \text{driving term.} \end{aligned} \tag{3.19}$$

The damping parameter λ corresponds in our notation to $\mathrm{e}^{-\eta}$ while the noise terms ψ_t and η_t satisfy $\langle\eta_t\rangle = \langle\psi_t\rangle = 0$ and

$$\begin{aligned} \langle\eta_t\eta_{t'}\rangle &= |p_{t-1}|\hbar\lambda(1-\lambda)\delta_{t,t'} \\ \langle\psi_t\psi_{t'}\rangle &= \frac{1}{4|p_{t-1}|}\hbar\frac{(1-\lambda)}{\lambda}\delta_{t,t'} \\ \langle\eta_t\psi_{t'}\rangle &= 0. \end{aligned} \tag{3.20}$$

This characterization of these noise terms should be compared with our result (3.16) for the noise terms $F(t)$ and $G(t)$. The map (3.19) has been obtained by DG from the full quantal propagator after applying the master-equation formalism [9] and taking the 'semiclassical' limit. The significance of these approximations should be clarified. The master-equation approach is based on a Markovian treatment of the dynamics. Thus, noise time-autocorrelations that may arise are automatically ignored and therefore we cannot tell whether the noise terms in (3.19) are white due to some special feature of the model or due to the Markovian approximation involved. By 'taking the semiclassical limit', DG meant that the condition $\mathcal{O}(\hbar) \ll \min(\lambda, 1-\lambda)$ should be satisfied. This condition implies in my view that the discretization of the momentum variable $p = \hbar n$ is fine enough to support classical structures that are affected by the damping. By inspection of (3.19) it is observed that the

relevant classical scales after one iteration are $\mathcal{O}(\lambda)$ and $\mathcal{O}(1-\lambda)$ for either the residual or the change in the momentum, respectively. DG have observed that the noise terms ψ_t and η_t are $\hbar$-dependent and disappear in the limit of $\hbar \to 0$. Their conclusion has been that, regardless of the quantization scheme, 'in the semiclassical limit dissipative quantum maps reduce to the classical maps with additional Gaussian noise terms determined by quantum theory.' However, it is evident that the procedure which has been adopted to obtain their map does not enable one to distinguish between quantal effects that are due to the quantum nature of the zero(!)-temperature bath and those quantal effects that are associated with the quantal nature of the rotator itself. The reason is that their 'semiclassical' limit has been taken *after* the elimination of the bath degrees of freedom. Furthermore, the DG approach does not permit comparison with the classical limit since the latter is not explicit in their model. In the present work, the classical limit of the rotator has been studied explicitly, leading to the conclusion that (3.19) could be obtained by treating the rotator as a classical object while the bath is quantum mechanical. A comparision between the two approaches—the DG approach versus our approach—is illustrated in figure 1.

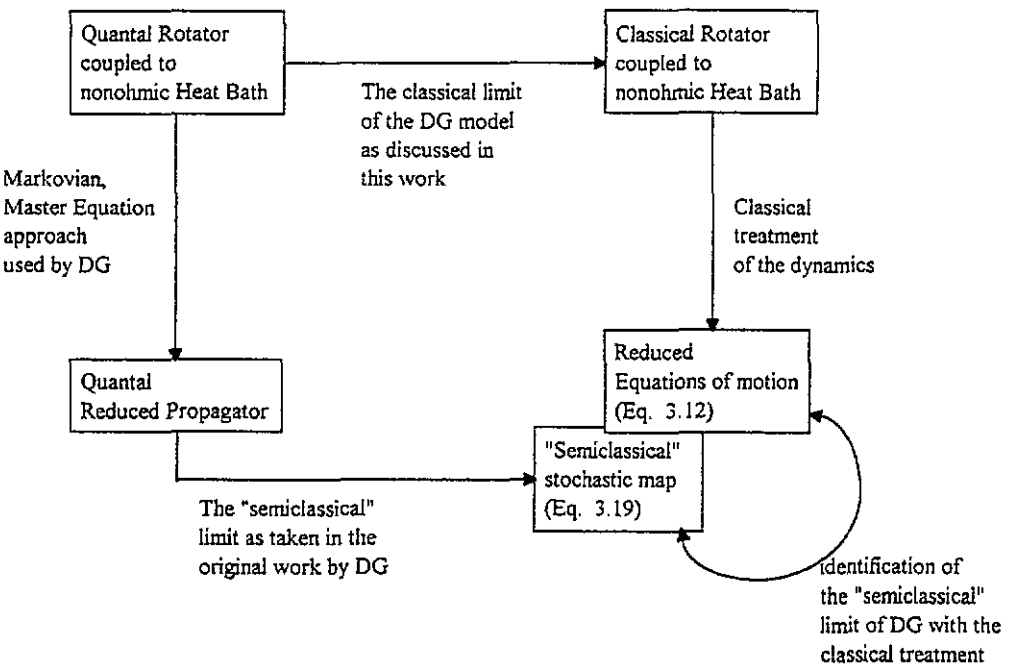

Figure 1. Illustration of the relation between the DG results and the classical limit.

4. Diffusion in the QKR problem

We consider a rotator whose unperturbed Hamiltonian is

$$\mathcal{H}_0 = \tfrac{1}{2}\hat{p}^2 + K\cos\hat{x}\sum_{n=-\infty}^{\infty}\delta(t-n). \tag{4.1}$$

Note that K and $\hbar$ are the dimensionless parameters of the Hamiltonian since the mass of the rotator, the spatial periodicity of the kicking potential and its time period are fixed in a natural way. The total Hamiltonian of the rotator plus the bath is (2.7). If the dynamics of the rotator are treated classically then the Langevin equations (3.12) apply. Integration of these equations over one time period (damped propagation + kick) yields the stroboscopic map

$$x(t) = x(t-1) + \frac{1}{\eta}(1 - \mathrm{e}^{-\eta})p(t-1) + g(t)$$

$$p(t) = \mathrm{e}^{-\eta}p(t-1) + K\sin x(t) + f(t) \tag{4.2}$$

where t will denote a discrete (integer) time variable from now on. The noise terms are given by

$$f(t) = \int_{t-1}^{t} \mathrm{e}^{-\eta(t-\tau')}F(\tau')\,\mathrm{d}\tau' \tag{4.3}$$

and

$$g(t) = \int_{t-1}^{t}\int_{t-1}^{\tau''} \mathrm{e}^{-\eta(\tau''-\tau')}F(\tau')\,\mathrm{d}\tau'\,\mathrm{d}\tau''. \tag{4.4}$$

Note that for the ohmic model $g(t)$ does not include an $F(t)$-independent component since a $G(t)$ term, as in (3.12), is absent. For the sake of later convenience we define the discrete-time autocorrelation function

$$\nu(t-t') \equiv \langle f(t)f(t')\rangle \tag{4.5}$$

and denote the variance $\nu(0)$ of the noise by ν. The general expression for $\nu(t-t')$ in the ohmic model is

$$\nu(t-t') = \int_{t-1}^{t}\int_{t'-1}^{t'} \mathrm{e}^{-\eta(t-\tau+t'-\tau')}\phi_{\mathrm{ohmic}}(\tau-\tau')\,\mathrm{d}\tau\,\mathrm{d}\tau'. \tag{4.6}$$

In the weak-damping regime ($\eta t \ll 1$), one may use the simpler relation

$$\nu(t-t') = \int_{t-1}^{t}\int_{t'-1}^{t'} \phi_{\mathrm{ohmic}}(\tau-\tau')\,\mathrm{d}\tau\,\mathrm{d}\tau'. \tag{4.7}$$

For the sake of later comparison, we note that similar considerations in the simplified DG model also yield, after inessential simplification, the map (4.2). The noise term $f(t)$ then has, in the weak damping regime, the discrete-time autocorrelation function

$$\nu_{\mathrm{DG}}(t-t') = \langle|p|\rangle\int_{t-1}^{t}\int_{t'-1}^{t'} \phi_{\mathrm{DG}}(\tau-\tau')\,\mathrm{d}\tau\,\mathrm{d}\tau' \tag{4.8}$$

instead of (4.7). The noise variable $g(t)$ includes, in the latter case, two contributions: one is a spreading term analogous to (4.4) and the other is the discrete-time version of the noise variable $G(t)$ that appears in equation (3.12). If the dynamics take place far enough from $|p| = 0$ then the $G(t)$ term is negligible.

Focusing on the weak damping regime, explicit expressions for the noise autocorrelation function $\nu(t-t')$ are found by substitution of results (2.28) and (3.18) into (4.7) and (4.8), respectively. In the zero-temperature DG model the noise is white and its variance depends on the region in momentum space where the dynamics take place, namely

$$\nu_{\mathrm{DG}} = \hbar\eta\langle|p|\rangle. \tag{4.9}$$

At finite temperature, $\nu = \infty$ due to the 1/frequency component (3.17) of the noise-fluctuation power spectrum. In the ohmic model, on the other hand, the noise is homogeneous in phase space and is controlled by the temperature. At high temperatures the noise is white with

$$\nu = 2\eta/\beta \qquad \text{for } \hbar\beta \ll 1. \tag{4.10}$$

As the temperature is lowered, ν decreases. But when $\hbar\beta$ becomes larger than the period of the kicking (which is 1 in our system of units) the variance stops decreasing and acquires a cutoff-dependent logarithmic term, namely

$$\nu = \frac{2\pi}{6}\hbar\eta + 2\frac{\hbar\eta}{\pi}\ln\tilde{\omega}_{\mathrm{c}} \qquad \text{for } 1 \ll \hbar\beta. \tag{4.11}$$

Details of computation are presented in appendix C. The parameter $\tilde{\omega}_{\mathrm{c}}$ is related there to ω_{c}. The noise possesses, in the latter case, long-range autocorrelations

$$\nu(t-t') = \begin{cases} -\dfrac{\hbar\eta}{\pi}(1+\ln\tilde{\omega}_{\mathrm{c}}) & \text{for } |t-t'| = 1 \\ -\dfrac{\hbar\eta}{\pi}\dfrac{1}{|t-t'|^2} & \text{for } 1 < |t-t'| \ll \hbar\beta \\ -2\dfrac{\eta}{\beta}\dfrac{2\pi}{\hbar\beta}\exp\left(-\dfrac{2\pi}{\hbar\beta}|t-t'|\right) & \text{for } \hbar\beta \ll |t-t'|. \end{cases} \tag{4.12}$$

Within the framework of the Markovian treatment, the autocorrelation function is taken to be

$$\nu_{\text{Markovian treatment}}(t-t') \equiv \nu\delta_{t,t'}. \tag{4.13}$$

Thus, if the noise is not white, as in the case of the low-temperature ohmic model, its autocorrelations are ignored in this approximation. If the system is chaotic and is treated classically, the neglecting of noise time-autocorrelations is justified by the exponential instability of the phase-space trajectories. In this case we expect no *memory* for long-range noise autocorrelations. For the quantum mechanical problem the situation may be quite different due to long-range *dynamical* correlations. The effect of the latter will be discussed later in this section.

In what follows we are interested in the time evolution of the momentum-dispersion function, namely

$$E(t) = \langle\langle(p(t)-p(0))^2\rangle\rangle$$

where $\langle\langle\ \rangle\rangle$ denotes here the uniform statistical average over initial conditions. The diffusion coefficient is then

$$\mathcal{D} = \lim_{t\to\infty}(E(t+1)-E(t)). \tag{4.15}$$

In the absence of noise and friction as the value of K is increased, the *classical* dynamics that is generated by the map (4.2) follow the KAM scenario [1]. For $K < K_c$ ($K_c \simeq 0.9716$) diffusion is impossible ($\mathcal{D} = 0$) due to the existence of KAM curves. For $K_c < K$ the last KAM trajectories that bound diffusion in momentum space have already been destroyed. For large enough K (say $5 < K$) a reasonably good approximation for the diffusion coefficient is

$$\mathcal{D}_{\text{classical}} = [1 - 2\mathcal{J}_2(K)]\tfrac{1}{2}K^2 \tag{4.16}$$

where $\mathcal{J}_2$ denotes the Bessel function of order two. The leading term ($\mathcal{D} = \frac{1}{2}K^2$) may be obtained from the map (4.2) by assuming that successive values of the variable x are uncorrelated. The deviations from this value constitute a manifestation of dynamical correlations. The latter become negligible for 'harder' chaos ($1 \ll K$). The effect of noise on classical diffusion has been studied by Karney *et al* [17]. If the noise is strong ($\nu \sim 1$) then dynamical correlations are destroyed and the expression $\mathcal{D} = \frac{1}{2}K^2$ becomes exact. Furthermore, if ν is of the order K^2, or larger, then enhanced diffusion with coefficient $\mathcal{D} = \frac{1}{2}K^2 + \nu$ is attained. The diffusion in momentum space may be described by a Fokker–Planck equation [1]. Taking into account the weak-damping effect, the time evolution of the momentum-distribution function $\rho(p)$ is determined by

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial p^2}(\mathcal{D}\rho) + \frac{\partial}{\partial p}(\eta p \rho). \tag{4.17}$$

The immediate result of the Fokker–Planck equation is that diffusion in momentum is suppressed on a time scale $t_r \equiv 1/\eta$. For $t_r \ll t$, a steady state is reached

$$\rho_\infty(p) = \left(\frac{\eta}{\pi \mathcal{D}}\right)^{1/2} \mathrm{e}^{-(\eta/\mathcal{D})p^2} \tag{4.18}$$

and thus

$$\langle\langle p^2 \rangle\rangle_\infty = \mathcal{D}/\eta. \tag{4.19}$$

We turn to the case of strong damping ($1 < \eta$). Inspection of the map (4.2) reveals that strong damping tends to attract the classical trajectory to the curve $p = \sin x$ leading to a strange attractor [18]. However, the effect of noise is to smear the fine structures of this attractor [8].

So far the rotator has been treated as a classical object. We turn now to analyse the quantum mechanical time evolution. As a first step let us define the parameter regime which is of special interest. The case of strong dissipation ($1 < \eta$) is not of great interest since the steady state is achieved within several time steps and quantum-to-classical correspondence is expected to hold. Indeed, DG have introduced numerical evidence that, in the limit of strong dissipation of the damped QKR problem, their semiclassical map (3.19) 'suffices to reproduce all observable effects' [8]. However, we have demonstrated that, in essence, the *same* map may be obtained by treating the rotator as a classical object. We therefore conclude that classical treatment of the dynamics is satisfactory for any practical purpose in the case of strong damping.

From now on we focus our discussion on the weak-damping regime ($\eta \ll 1$) of the damped QKR. Here the damping effect is negligible on the time scale $t \ll t_r$ and therefore to determine the diffusion in momentum space it is legitimate to replace the bath by an

equivalent c-number noise source as in (2.43). The one-step propagator that corresponds to the classical map (4.2) within this time domain is

$$\hat{U} = \mathcal{T}\exp\left[-\mathrm{i}\int_{t-1}^{t}(\mathcal{H}_0 - \int \mathrm{d}\varphi\, F_\varphi\sqrt{2}\sin(\hat{x}+\varphi))\,\mathrm{d}\tau\right]. \tag{4.20}$$

The operation $\mathcal{T}\exp$ denotes time-ordered exponentialization. An explicit expression for U can be obtained only in the absence of noise. In what follows we shall adopt the conventional approach: the operator U is to be approximated by its discrete-time version

$$\hat{U} = \exp\left[-\frac{\mathrm{i}}{\hbar}(K\cos\hat{x} + \hat{V}_{\mathrm{int}})\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{1}{2}\hat{p}^2\right] \tag{4.21}$$

with an interaction term that is assumed to be effective only during the kick process. Thus

$$\hat{V}_{\mathrm{int}} = \int \mathrm{d}\varphi\, f_\varphi(t)\sqrt{2}\sin(\hat{x}+\varphi) \tag{4.22}$$

where $f_\varphi(t)$ are real functions of the discrete-time variable t that satisfy upon averaging $\langle f_\varphi(t)\rangle = 0$ and

$$\langle f_\varphi(t) f_{\varphi'}(t')\rangle = \frac{1}{2\pi}\delta(\varphi-\varphi')\int_{t-1}^{t}\int_{t'-1}^{t'}\phi(\tau-\tau')\,\mathrm{d}\tau\,\mathrm{d}\tau'. \tag{4.23}$$

One may wonder whether any physics is missed by switching from the exact expression (4.20) to its discrete-time version (4.21). Indeed, the discrete-time version does not take into account the spreading effect: it is easily found that the classical map that corresponds to (4.21) is the same as (4.2) except that the noise term $g(t)$ is absent. The latter is associated with the noise term $f(t)$ via spreading in the x-direction. The insignificance of this spreading in the QKR problem has been discussed in section 5 of [10]. We therefore consider (4.21) to be a reasonable approximation for the true propagator.

Quantum mechanically, in the absence of noise and friction, the classical diffusive behaviour is suppressed ($\mathcal{D}_{\mathrm{quantal}} = 0$) [2]. In order to explain this effect it has been argued [3] that the eigenstates $|r\rangle$ of the one-step propagator $\hat{U}$ are localized in the p-representation with localization length $\hbar\xi$ which is given by $\xi \simeq \frac{1}{2}D_0/\hbar^2$ [19]. Here D_0 denotes the initial diffusion rate. It may be estimated [19] by using the classical result (4.16) with K replaced by $(2\sin\hbar/2)(K/\hbar)$. The eigenvalues of $\hat{U}$ are denoted by $\mathrm{e}^{-\mathrm{i}\omega_r}$ where ω_r are the quasi-energies. Assuming that the rotator is prepared in momentum eigenstate $|p\rangle$, it follows that the quantum state of the rotator is approximately a superposition of ξ quasi-energy eigenstates $|r\rangle$. For short time $t < t^*$ ($t^* \sim 2\xi$), a classical-like diffusive behaviour is followed, but on larger time scales the dynamics appears to be quasiperiodic. For the sake of completeness, we note that the proper definition of the energy function is now

$$E(t) \equiv \overline{\langle p|(\hat{p}(t)-p)^2|p\rangle} \tag{4.24}$$

where the bar denotes the uniform statistical average over the states $|p\rangle$. The latter expression can be written in the form (4.14) to emphasize that a particular representation is not essential. From now on the notation $\langle\langle\ \rangle\rangle$ stands for quantum statistical average, i.e.

$$\langle\langle\hat{o}\rangle\rangle = \lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\,\mathrm{trace}(\hat{o}) \tag{4.25}$$

where $\mathcal{N}$ denotes the dimension of the basis that is used to compute the trace. Expression (4.14) reduces to (4.24) if the momentum representation is used in this computation. It also corresponds to the classical statistical average in the semiclassical limit.

The introduction of weak noise into the QKR system results in the destruction of localization. Using a heuristic picture Ott, Antonsen and Hanson [4] have argued that the coherence time is simply the period it takes for the noise to 'mix' neighbouring momenta. The diffusion that is induced by the noise (disregarding the interplay with the kicks) is $\delta p^2(t) = \nu t$ and the condition $\delta p \simeq \hbar$ for destruction of coherence leads to the result $t_c = \hbar^2/\nu$. Within the framework of an analytical approach, a satisfactory definition of the coherence time should be $t_c \equiv \Gamma^{-1}$, where Γ denotes the (average) decay constant of the quasi-energy eigenstates. A formal approach to analyse the decay process has been presented by the author [10]. A generalization of this approach for the ohmic model will be outlined briefly in the present paragraph. This generalization takes into account that the noise is not necessarily white. A first-order perturbative estimate for the transition probability from state $|r\rangle$ to state $|s\rangle$ after time t is

$$\mathrm{PROB}(s|r) = \sum_{\tau=1}^{t}\sum_{\tau'=1}^{t} e^{i\omega_{sr}(\tau-\tau')}\langle s|\frac{V_{\mathrm{int}}(\tau)}{\hbar}|r\rangle^*\langle s|\frac{V_{\mathrm{int}}(\tau')}{\hbar}|r\rangle \tag{4.26}$$

where ω_{sr} denote differences in quasi-energies. Note that the Schrödinger picture is used here. Substituting (4.22), and averaging over realizations of the noise, one obtains the transition probability per unit time

$$W(s|r) = \left[\int_0^{2\pi}\frac{d\varphi}{2\pi}\left|\langle s|\frac{\sqrt{2}\sin(\hat{x}+\varphi)}{\hbar}|r\rangle\right|^2\right]\nu(\omega_{sr}) \tag{4.27}$$

where $\nu(\omega) \equiv \sum_{\tau=-\infty}^{\infty}\nu(\tau)e^{i\omega\tau}$ is the noise-fluctuation spectrum. The average decay rate is

$$\Gamma \equiv \lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\sum_{rs}W(s|r). \tag{4.28}$$

This expression can be cast into the form

$$\Gamma = \left(\frac{1}{\hbar}\right)^2\int_{-\pi}^{\pi}\frac{d\omega}{2\pi}C(\omega)\nu(\omega) \tag{4.29}$$

where

$$C(\omega) \equiv \int_0^{2\pi}\frac{d\varphi}{2\pi}\lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\sum_{rs}\left|\langle s|\sqrt{2}\sin(\hat{x}+\varphi)|r\rangle\right|^2 2\pi\delta(\omega-\omega_{sr}). \tag{4.30}$$

An alternative way to present the latter result is

$$\Gamma = \frac{1}{\hbar^2}\sum_{\tau=-\infty}^{\infty}C(\tau)\nu(\tau) \tag{4.31}$$

where $C(\tau) = C_s(\tau) + C_c(\tau)$ with the real symmetric functions

$$\begin{aligned} C_s(\tau) &\equiv \langle\langle \sin\hat{x}(\tau)\sin\hat{x}(0)\rangle\rangle \\ C_c(\tau) &\equiv \langle\langle \cos\hat{x}(\tau)\cos\hat{x}(0)\rangle\rangle. \end{aligned} \tag{4.32}$$

The interaction picture is used in later definitions. It is easily verified that $C(\tau = 0) = 1$. Thus, for white noise, one immediately obtains

$$\Gamma = \frac{1}{\hbar^2}\nu \tag{4.33}$$

and therefore the heuristic result $t_c = \hbar^2/\nu$ is formally recovered. However, if the noise is not white then knowledge of the global behaviour of $C(\tau)$ is needed. Incidentally $C(\tau = 1) = 0$ and there are indications [19] that the other first few correlations are given approximately by the classical expressions with K replaced by $2\sin(\hbar/2)(K/\hbar)$. Classically, the long-time correlations drop to zero due to the exponential instability of the phase-space trajectories. It has been demonstrated quantum mechanically [10] that dynamical correlations decay only after a relatively long time period t^*. The slow decay of dynamical correlations is a manifestation of the localization effect that is responsible for the suppression of chaos in the QKR system. We shall assume, based on the results of a preceding study [10], that the following expression holds:

$$C(\tau) \simeq -\frac{c}{2t^*}\exp\left(-\frac{|\tau|}{t^*}\right) \qquad \text{for } 1 < |\tau| \tag{4.34}$$

where c is of order unity. We turn now to perform an explicit estimate of Γ. At high temperatures the noise is white, its variance is given by (4.10) and (4.33) applies. Hence

$$\Gamma = \frac{\eta}{\hbar}\left(\frac{2}{\hbar\beta}\right) \qquad \text{for } \hbar\beta \ll 1. \tag{4.35}$$

At very low temperatures one should substitute (4.11)–(4.12) and (4.34) into (4.31) and perform the summation (appendix D). The result is

$$\Gamma = \frac{\eta}{\hbar}\left[c_1 + c_2 \ln\tilde{\omega}_c + \frac{c}{t^*}\left(c_3 - \frac{1}{t^*}(c_4 + \ln t^*)\right)\right] \tag{4.36}$$

where all the c's are of order unity, namely $c_1 = (\pi/3)$, $c_2 = (2/\pi)$, $c_3 = (\pi^2/6) - 1$ and $c_4 \simeq 0.4$. This expression includes three terms. The third is a manifestation of the interplay between noise time-autocorrelations and dynamical correlations. By setting c to zero, one obtains the Markovian approximation. In the case of the zero-temperature DG model, the noise is white. Substitution of (4.9) into (4.33) leads to the result

$$\Gamma = \frac{\eta}{\hbar}\langle|p|\rangle \qquad \text{for the DG model.} \tag{4.37}$$

Here the decay rate is different for eigenstates that are located in various regions of momentum space. In particular, for those eigenstates that are in the vicinity of the origin ($|p| = 0$), the expression $\Gamma \sim \eta\xi$ roughly holds, while for eigenstates that are located around a distant site $p = \hbar n_0$ with $\xi \ll |n_0|$ the expression $\Gamma \simeq \eta n_0$ applies.

We are now ready to discuss the diffusion that is induced by the noise. Following the heuristic picture that has been presented by Ott *et al* [4] we distinguish three noise regimes. For weak noise ($t^* \ll t_c$), the diffusion process in momentum space is similar to a random walk on a grid with spacing $\hbar\xi$ and hopping probability Γ. The diffusion coefficient in the presence of weak noise is therefore $\mathcal{D} \simeq (\hbar\xi)^2(1/t_c)$. If the noise is not weak ($t_c < t^*$) then classical-like diffusion is recovered [4] and $\mathcal{D} \simeq D_0$. For even larger noise intensity

this diffusion is enhanced [4] and is given by $D = \frac{1}{2}K^2 + \nu$ as in the classical case. A formal analytical treatment of the diffusion process has been presented elsewhere by the author [10]. In the weak noise regime it has been established that

$$\mathcal{D} = t^* D_0 \Gamma = \frac{t^*}{t_c} D_0. \tag{4.38}$$

Using the relation $t^* \sim 2\xi \sim D_0/\hbar^2$, one recovers the heuristic result up to a prefactor that has not been determined in the original version. The decay rate Γ and the associated coherence time t_c should be determined via equation (4.31). Thus, the derivation that led to (4.38) has proved that diffusion is affected by noise time-autocorrelations. Numerical experiments to verify this conclusion and the validity of equation (4.38) have been performed [15, 10]. In the ohmic model, substitution of either (4.35) or (4.36) into (4.38) leads to

$$\mathcal{D} = \begin{cases} C\dfrac{K^4}{\hbar^4}\nu & \text{for } \nu \ll \dfrac{\hbar^4}{K^2} \\ D_0 & \text{for } \dfrac{\hbar^4}{K^2} \ll \nu \ll K^2 \\ \dfrac{1}{2}K^2 + \nu & \text{for } K^2 \ll \nu \end{cases} \tag{4.39}$$

where ν is a function of the temperature (see (4.10)–(4.12)). The factor C is of order unity for white noise and is slightly larger (but still of order unity) at low temperatures due to the noise time-autocorrelations. In the latter case, the factor includes the ratio of Γ, as given by (4.36), to the same expression with $c = 0$. In comparing the ohmic model with the DG model, we also introduce the explicit expression for $\mathcal{D}$ for the latter case. Substitution of (4.37) into (4.38) yields

$$\mathcal{D}(p) = \begin{cases} C_1\dfrac{K^6}{\hbar^4}\eta & \text{for } |p| \ll \dfrac{K^2}{\hbar} \\ C_2\dfrac{K^4}{\hbar^3}\eta|p| & \text{for } \dfrac{K^2}{\hbar} \ll |p| \ll \dfrac{\hbar^3}{K^2}\dfrac{1}{\eta} \\ D_0 & \text{for } \dfrac{\hbar^3}{K^2}\dfrac{1}{\eta} \ll |p| \ll \dfrac{K^2}{\hbar}\dfrac{1}{\eta} \\ \dfrac{1}{2}K^2 + \hbar\eta|p| & \text{for } \dfrac{K^2}{\hbar}\dfrac{1}{\eta} \ll |p| \end{cases} \tag{4.40}$$

where C_1 and C_2 are constants of order unity. Here $\mathcal{D}$ is momentum-dependent leading, in general, to a superdiffusive behaviour. The weak-noise regime $|p| < \hbar^3/K^2(1/\eta)$ is effectively absent if its width is less then the localization length. Therefore, for strong coupling ($(1/\xi)^2 < \eta$) the diffusion is essentially classical.

Eventually, we should discuss the steady state in the case of weak damping. The phenomenological Fokker–Planck equation (4.17) should be valid whenever a stochastic picture of the diffusion process applies. This is evidently the case if the noise is strong ($\mathcal{D} \simeq \mathcal{D}_{\text{classical}}$). But this is also the case if the noise is weak provided $t_c \ll t_r$. The most difficult situation in which this latter condition should be satisfied is when the temperature is very low. It is easily found that a sufficient condition for $t_c \ll t_r$ to hold is $\hbar \ll 1$. Whenever the Fokker–Planck equation holds, quantal effects enter only via the diffusion process and the nature of the steady state is determined accordingly. For strong noise,

$\mathcal{D} \simeq \mathcal{D}_{\text{classical}}$ and therefore the quantal steady state corresponds to the classical steady state. If, on the other hand, the noise is weak, then $\mathcal{D} \ll \mathcal{D}_{\text{classical}}$ and consequently the quantal steady state does not correspond to the classical steady state. The investigation of this state is left for future studies.

5. Conclusions

Two models for the investigation of the damped quantum kicked rotator (QKR) problem have been presented. The first model constitutes a generalization of the CL model for a damped particle, to the case of the damped rotator, where the coupling of the bath degrees of freedom is to the angle variable of the rotator. The second model is a simplification of the DG model. The simplified DG model enables one to study the classical limit of the original DG model.

The ohmic model yields complete correspondence with the classical Langevin equation for a damped rotator (2.22). In this model there appears a *friction* term (2.27) that is proportional to velocity irrespective of the detailed coupling scheme which is purely position-dependent and *noise* which is white at high temperatures but exhibits long-range time-autocorrelations at the limit of zero temperature (2.28). The DG model and its simplification yield only partial correspondence to the classical Langevin equation. Its classical limit is represented by equation (3.12). There appears an ohmic-like damping term (3.14) due to the particular dependence of the coupling scheme to the bath (which is not ohmic) on the momentum variable. However, there is also an anomalous friction term (3.15) that diverges in the limit $p \to 0$. The noise in the DG model and its simplification (3.16) is inhomogeneous in momentum space, uncorrelated at zero temperature (3.18) and turns out to be 1/frequency noise (3.16) at finite temperatures. These latter features make the DG model and its simplification appear to be of less physical relevance compared with the ohmic model.

In the strong-damping case, it has been shown via re-interpretation of the DG results that a classical treatment of the dynamics for the rotator is also sufficient in the quantum mechanical problem. We arrived at this conclusion (see figure 1 for illustration) by demonstrating that the semiclassical stochastic map of DG may be derived using classical equations of motion for the rotator. This point was not clear from the original analysis of DG since the limit of strong damping was taken only after the quantal propagator had been computed, giving the impression that the quantal treatment is essential. The steady state, in the case of strong damping, is reached after a few time steps and on this time scale there is correspondence with the classical behaviour. The fine details of the classical strange attractor are smeared due to the noise.

In the weak-damping case, the steady state is reached only after a relatively long relaxation time. On shorter time scales damping is insignificant and the bath may be replaced by an equivalent c-number noise source (2.43). For strong noise, classical diffusion is recovered and therefore the interesting regime is that of weak noise. If the noise is weak, namely $t^* \ll t_c$, where t^* is the breaktime and t_c is the coherence time, then destruction of localization may be treated within the framework of perturbation theory. One may determine the average decay rate Γ of the eigenstates (4.31) and thus the induced diffusion (4.38). Indeed, explicit expressions have been obtained for the decay constant (equations (4.35) and (4.36)) and for the diffusion coefficient (4.39), in the case of the ohmic model, and compared with the corresponding results for the simplified zero-temperature DG model (equations (4.37) and (4.40)). A Markovian treatment of the dynamics is found to be exact

for white (uncorrelated) noise. At low temperatures the ohmic noise possesses long-range time-autocorrelations (4.12) and it is found that the Markovian treatment in the latter case underestimates the induced decay (equation (4.36) with $0 < c$) and therefore the associated diffusion (via equation (4.38)). This is quite different from what is found in the case of either an undriven particle or an undriven rotator coupled to an ohmic bath [12, 5, 14, 10] or even in the case of QKR coupled to an ohmic bath via its momentum variable [14]. For all those examples it has been found that diffusion is either suppressed or reduced significantly due to the noise time-autocorrelations at zero temperature. In the present case of damped QKR, the effect of low-temperature noise autocorrelations is relatively small and of opposite trend, i.e. diffusion is *enhanced* (equation (4.38) with (4.36) where $0 < c$) but not significantly. The origin for the dissimilar manifestations of noise time-autocorrelations in the case of different dynamical systems is the difference in the dynamical correlations that are involved. Destruction of coherence is determined by the interplay of noise autocorrelations with the dynamical correlations. The latter are specific for each system and coupling involved.

The relaxation towards a steady state in the case of weak damping has been discussed using the phenomenological Fokker–Planck equation (4.17). A steady state is reached when diffusion is balanced by friction. Quantal effects manifest themselves only via the diffusion process. For large noise, the steady state is classical-like, whereas in the case of weak noise it does not correspond to the classical steady state. The complete understanding of the relaxation process is left as an open problem for further study. The validity of the Fokker–Planck equation is not justified if the coherence time is not much shorter than the relaxation time. Furthermore, it would be nice to find a formalism (e.g. some modification of the Feynman–Vernon formalism) that will enable us to find explicitly the non-classical steady state that is reached due to the damping process.

Acknowledgments

I thank S Fishman for useful criticism and fruitful guidance. This work was supported in part by the US–Israel Binational Science Foundation (BSF).

Appendix A. The Wigner function for the harmonic oscillator at equilibrium

The probability density matrix for a quantal system whose Hamiltonian is $\mathcal{H}(x, p)$ at canonical thermal equilibrium is

$$\rho_{\rm eq}(x'', x') = \frac{1}{\mathcal{Z}}(x''|{\rm e}^{-\beta\mathcal{H}}|x') \tag{A.1}$$

where β is the reciprocal temperature $(1/k_{\rm B}T)$ and $\mathcal{Z} \equiv {\rm trace}({\rm e}^{-\beta\mathcal{H}})$ is the partition function. For a harmonic oscillator, namely $\mathcal{H} = p^2/2m + \frac{1}{2}m\omega^2x^2$, one obtains

$$\rho_{\rm eq}(x'', x') = \sqrt{\frac{m\omega}{2\pi\hbar\sinh(\beta\hbar\omega)}}\exp\left[\frac{-m\omega}{2\hbar\sinh(\beta\hbar\omega)}((x''^2 + x'^2)\cosh(\beta\hbar\omega) - 2x''x')\right]. \tag{A.2}$$

This result may be derived using a variety of techniques [20].

The Wigner function is an optional representation of a quantal state. It is defined as follows:

$$\rho(x,p)=\int_{-\infty}^{\infty}\rho(x+\tfrac{1}{2}r,x-\tfrac{1}{2}r)\mathrm{e}^{-\mathrm{i}pr/\hbar}\,\mathrm{d}r. \tag{A.3}$$

It is a real function that satisfies $\int\int \mathrm{d}x\,\mathrm{d}p/(2\pi\hbar)\rho(x,p)=1$. Integration of $\rho(x,p)$ over either x or p yields, respectively, the probability distribution function of either the p or the x variable. For the harmonic oscillator at thermal equilibrium one obtains the following explicit expression:

$$\rho_{\mathrm{eq}}(x,p)=\frac{1}{\frac{1}{2}\coth(\frac{1}{2}\beta\hbar\omega)}\exp\left[-\beta\left(\frac{\tanh(\frac{1}{2}\beta\hbar\omega)}{\frac{1}{2}\beta\hbar\omega}\right)\left(\frac{p^2}{2m}+\tfrac{1}{2}m\omega^2x^2\right)\right]. \tag{A.4}$$

This result reduces to the classical limit provided that the temperature is large enough compared with the energy quantization, namely $\hbar\omega\ll k_{\mathrm{B}}T$.

Appendix B. The reduced equations of motion for the simplified DG model

Starting with the full Hamiltonian (2.8) with the interaction term (3.10), one obtains the following equation of motion:

$$\begin{aligned}\dot{x}&=p-\sum_{\alpha}C_\alpha q_\alpha\sqrt{2}\sin(x+\varphi_\alpha)\frac{\mathrm{sgn}(p)}{2\sqrt{|p|}}\\ \dot{p}&=-V'(x)+\sum_{\alpha}C_\alpha q_\alpha\sqrt{2}\cos(x+\varphi_\alpha)\sqrt{|p|}\end{aligned} \tag{B.1}$$

while $q_\alpha(t)$ satisfies (2.11) with the replacement $C_\alpha\to C_\alpha\sqrt{|p|}$. Using the same definition (2.10) for $\mathcal{F}_\varphi(t)$, one obtains, instead of (2.9), the following equations

$$\begin{aligned}\dot{x}&=p-\int_0^{2\pi}\mathrm{d}\varphi\,\sqrt{2}\sin(x+\varphi)\frac{\mathrm{sgn}(p)}{2\sqrt{|p|}}\mathcal{F}_\varphi(t)\\ \dot{p}&=-V'(x)+\int_0^{2\pi}\mathrm{d}\varphi\,\sqrt{2}\cos(x+\varphi)\sqrt{|p|}\mathcal{F}_\varphi(t).\end{aligned} \tag{B.2}$$

In order to eliminate the implicit dependence of (B.2) via $\mathcal{F}_\varphi(t)$ on the bath degrees of freedom, one should solve the equation of motion for $q_\alpha(t)$ as in (2.12), substitute into (2.10) and introduce the result into (B.2). One then obtains the following explicit expressions for the friction terms:

$$\begin{aligned}F^{\mathrm{friction}}&=-\int_0^t 2\alpha(t-t')\sqrt{|p(t)p(t')|}\sin[x(t)-x(t')]\,\mathrm{d}t'\\ G^{\mathrm{friction}}&=-\int_0^t 2\alpha(t-t')\frac{\sqrt{|p(t)p(t')|}}{2p(t)}\cos[x(t)-x(t')]\,\mathrm{d}t'.\end{aligned} \tag{B.3}$$

The response kernel is defined as in (2.15) with the appropriate spectral function (3.11). The explicit expression is

$$\alpha(\tau) = \int_0^\infty \frac{\mathrm{d}\omega}{\pi} J(\omega) \sin(\omega\tau) = \frac{\eta}{\pi} \frac{\tau}{\tau^2 + \tau_c^2}. \tag{B.4}$$

It is assumed that $\tau_c = 1/\omega_c$ is much shorter than any other relevant time scale, thus the approximations $p(t) \simeq p(t')$ and $x(t) - x(t') \simeq p(t - t')$ may be used in (B.3). The results (3.14) and (3.15) then follow immediately.

The computation of the noise terms for the simplified DG model is quite straightforward. Equation (2.20) still holds provided that the appropriate autocorrelation function, namely $\phi_{DG}(t - t')$, is used. The latter is found by substitution of the non-ohmic spectral function (3.11) in (2.21). The second-order moments $\langle F(t)F(t')\rangle$, $\langle G(t)G(t')\rangle$ and $\langle F(t)G(t')\rangle$ of the appropriate terms in (B.2) may then be computed to obtain (3.16).

Appendix C. Zero-temperature noise in the damped QKR ohmic model

The variance ν of the noise in the damped ohmic model may be found by substitution of (2.28) into (4.6). For weak damping, in the zero-temperature limit, one obtains

$$\nu = \frac{\hbar\eta}{\pi} \int_0^1 \int_0^1 \frac{\tau_c^2 - (t' - t)^2}{(\tau_c^2 + (t' - t)^2)^2} \,\mathrm{d}t' \,\mathrm{d}t = \frac{\hbar\eta}{\pi} 2 \ln \sqrt{1 + \left(\frac{1}{\tau_c}\right)^2}. \tag{C.1}$$

The autocorrelations with $1 < \tau$ may be found in a similar way, i.e. by substitution of (2.29) into (4.7). The result is somewhat clumsy. However, for any practical calculation, the result may be approximated by the time-discrete version of the corresponding time-continuous expression (2.29), namely

$$\nu(\tau) \simeq -\frac{\hbar\eta}{\pi} \frac{1}{\tau^2} \qquad \text{for } 1 < \tau. \tag{C.2}$$

In order to find $\nu(\pm 1)$ one may use the identity $\sum_{-\infty}^{\infty} \nu(\tau) = 0$ that follows the sum rule (2.31). Using the equality $\sum_{\tau=1}^{\infty} 1/\tau^2 = \pi^2/6$ one obtains

$$\nu(\pm 1) = \frac{\hbar\eta}{\pi} \left[\left(\frac{\pi^2}{6} - 1\right) - \ln \sqrt{1 + \left(\frac{1}{\tau_c}\right)^2} \right]. \tag{C.3}$$

The above result may be represented in the form of equations (4.11) and (4.12) provided one uses the following definition

$$\tilde{\omega}_c = \mathrm{e}^{-\pi^2/6} \sqrt{1 + \left(\frac{1}{\tau_c}\right)^2}. \tag{C.4}$$

For $\tau_c \ll 1$ it leads to $\tilde{\omega}_c \simeq 0.2\omega_c$. The advantage of using expressions (4.11) and (4.12) stems from the fact that both the cutoff-dependent and the cutoff-independent components of $\nu(\tau)$ satisfy the sum rule $\sum_{-\infty}^{\infty} \nu(\tau) = 0$.

Appendix D. Decay rate in the limit of zero temperature

Consider the damped QKR ohmic model in the limit of zero temperature. Weak coupling is assumed. The decay rate Γ may be found by substitution of (4.11), (4.12) and (4.34) into (4.31). One obtains

$$\Gamma = \frac{1}{\hbar^2}\nu + c\frac{\eta}{\hbar}\frac{1}{t^*}\left(f\left(\frac{1}{t^*}\right) - 1\right). \tag{D.1}$$

The second term originates from the noise time-autocorrelations. The function $f(\lambda)$ is defined as follows.

$$f(\lambda) = \sum_{\tau=1}^{\infty} e^{-\lambda\tau}\frac{1}{\tau^2}. \tag{D.2}$$

The variable λ is assumed to be very small ($\lambda \ll 1$) since the breaktime t^* is typically large compared with unity. Since $C(|\tau| = 1) = 0$, it follows that one should omit the first term in (D.2) which leads to the substraction of one from $f(\lambda)$ in equation (D.1).

The sum (D.2) may be evaluated for $\lambda \ll 1$ using standard 'tricks'. Its second derivative with respect to λ is $\sum_{\tau=1}^{\infty} e^{-\lambda\tau}$ and therefore $f''(\lambda) \simeq 1/\lambda$. Also, its first deriviative may be approximated by an exponential integral leading to $f'(\lambda) \simeq \gamma + \ln\lambda$, where $\gamma = 0.577$ is Euler's constant. It follows that the function $f(\lambda)$ itself may be approximated by the expression

$$f(\lambda) = \frac{\pi^2}{6} + \lambda(\ln\lambda - \tilde{\gamma}) \tag{D.3}$$

where $\tilde{\gamma} \simeq 1 - \gamma \simeq 0.4$, while $\pi^2/6$ equals the sum (D.2) for $\lambda = 0$. Substitution of (D.3) and (4.11) into (D.1) leads to (4.36).

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